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Boltzmann Limit and Quasifreeness for a Homogenous Fermi Gas in a Weakly Disordered Random Medium

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Abstract We discuss some basic aspects of the dynamics of a homogenous Fermi gas in a weak random potential, under negligence of the particle pair interactions. We derive the kinetic scaling limit for the momentum distribution function with a translation invariant initial state and prove that it is determined by a linear Boltzmann equation. Moreover, we prove that if the initial state is quasifree, then the time evolved state, averaged over the randomness, has a quasifree kinetic limit. We show that the momentum distributions determined by the Gibbs states of a free fermion field are stationary solutions of the linear Boltzmann equation; this includes the limit of zero temperature.

Keywords Fermi gas · Quantum dynamics in random medium · Boltzmann limit · Quasifreeness

1 Introduction

We investigate the Boltzmann limit for the dynamics of a quantized field of non-relativistic electrons in a disordered medium. The analysis presented here is closely related to the derivation of Boltzmann equations from the quantum dynamics of the one-particle Anderson model at weak disorders, and involves techniques developed in [9–11, 13, 14] and [7, 8]; see also [19, 21]. We refer also to [1, 5, 6, 18, 20] for related works.

We consider a gas of fermions on the lattice $\Lambda_L := [-\frac{L}{2}, \frac{L}{2}]^d \cap \mathbb{Z}^d$ in dimension $d \geq 3$ and with periodic boundary conditions, for $L \gg 1$. We denote the dual lattice by $\Lambda_L^* = \Lambda_L/L$, and write $\int dp \equiv \frac{1}{L^d}\sum_{p\in\Lambda_L^*}$ for brevity. Letting $\mathfrak{F} = \bigoplus_{n\geq 0}\bigwedge_1^n \ell^2(\Lambda_L)$ denote the

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Fock space accounting for scalar fermions on Λ_L, we denote the creation- and annihilation operators by a_p^+, a_p, with $p \in \Lambda_L^*$, satisfying the usual canonical anticommutation relations.

Let $\mathfrak{A}$ denote the C^*-algebra of bounded operators on $\mathfrak{F}$. We let ρ_0 denote a translation invariant, normalized state on $\mathfrak{A}$, which preserves the particle number (i.e., $\rho_0(NA) = \rho_0(AN)$ for all $A \in \mathfrak{A}$, where $N = \sum_x a_x^+ a_x$ is the number operator).

We consider the Hamiltonian

$$H_\omega := \int dp E(p) a_p^+ a_p + \eta V_\omega \tag{1.1}$$

which generates the dynamics of a free Fermi gas coupled to a random potential

$$V_\omega := \sum_{x \in \Lambda_L} \omega_x a_x^+ a_x, \tag{1.2}$$

where $\{\omega_x\}_{x \in \Lambda_L}$ are real Gaussian i.i.d. random variables, and $0 < \eta \ll 1$ is a small coupling constant accounting for the disorder strength. We assume that the kinetic energy function is given by

$$E(p) = \sum_{j=1}^{d} \cos(2\pi p_j), \tag{1.3}$$

i.e., the Fourier multiplication operator determined by the centered nearest neighbor Laplacian $(\Delta f)(x) = \sum_{|y-x|=1} f(y)$ on $\mathbb{Z}^d$.

We are interested in the long-time dynamics of the fermion field described by

$$\rho_t(A) := \rho_0(e^{itH_\omega} A e^{-itH_\omega}), \tag{1.4}$$

where $A \in \mathfrak{A}$. While we are neglecting the pair interactions between the electrons, the effective interaction between the particles through their coupling to the random potential, and due to the Pauli principle remain significant. We prove the following. In a time scale $t = \frac{T}{\eta^2}$ where $T > 0$ denotes a macroscopic time variable, we find, in the thermodynamic limit, that for all $T > 0$ and for all test functions f, g of Schwartz class $\mathcal{S}(\mathbb{T}^d)$,

$$\Omega_T^{(2)}(f; g) := \lim_{\eta \to 0} \lim_{L \to \infty} \mathbb{E}[\rho_{T/\eta^2}(a^+(f) a(g))] = \int_{\mathbb{T}^d} dp F_T(p) \overline{f(p)} g(p), \tag{1.5}$$

where $F_T(p)$ satisfies the linear Boltzmann equation

$$\partial_T F_T(p) = 2\pi \int du\, \delta(E(u) - E(p))(F_T(u) - F_T(p)) \tag{1.6}$$

with initial condition $F_0(p) = \lim_{L \to \infty} \frac{1}{L^d} \rho_0(a_p^+ a_p)$. The proof is based on a generalization of methods due to Erdös and Yau in [11], and extended in [7], for the derivation of linear Boltzmann equations from the random Schrödinger dynamics in the weakly disordered 1-particle Anderson model.

We observe that if ρ_0 is the Gibbs distribution of the free fermion field, the corresponding momentum occupation density (the Fermi-Dirac distribution)

$$F_0(p) = \frac{1}{1 + e^{\beta(E(p) - \mu)}}, \tag{1.7}$$

for inverse temperature β and chemical potential μ, is a *stationary solution* of the linear Boltzmann equation (1.6), for all $\beta > 0$. This is also valid in the zero temperature limit $\beta \to \infty$ where in the weak sense,

$$\frac{1}{1+e^{\beta(E(p)-\mu)}} \to \chi[E(p) < \mu], \tag{1.8}$$

which is nontrivial if $\mu > 0$. Erdös, Salmhofer and Yau have proved in their landmark work [10, 13, 14] that for a time t beyond the kinetic scale η^{-2}, the effective dynamics of a single electron is *diffusive*; i.e., in this time scale, a wave packet evolves in position space according to the solution of a heat equation. Accordingly, we expect the Fermi-Dirac distribution to remain a stationary solution in the diffusive limit, and for the corresponding time scale addressed in [10, 13, 14].

We remark that the translation invariant model without the random potential (i.e., $\eta = 0$) but including the full repulsive particle pair interaction is determined by the Hamiltonian

$$\widetilde{H}_\lambda := \int dp E(p) a_p^+ a_p + \lambda \sum_{x,y\in\Lambda_L} a_y^+ a_x^+ v(x-y) a_x a_y. \tag{1.9}$$

It is widely believed that in a time scale $t = \frac{T}{\lambda^2}$, the momentum density $F_T(p) := \lim_{\lambda\to 0}\lim_{L\to\infty}\frac{1}{L^d}\rho_{T/\lambda^2}(a_p^+ a_p)$ for the dynamics generated by $\widetilde{H}_\lambda$ satisfies the *Boltzmann-Uhlenbeck-Uehling equation*

$$\begin{aligned}\partial_T F_T(p) = -4\pi \int & dp_1 dp_2 dq_1 dq_2 |\widehat{v}(p_1-q_1) - \widehat{v}(p_1-q_2)|^2 \delta(p-p_1)\\ & \times \delta(p_1+p_2-q_1-q_2)\delta(E(p_1)+E(p_2)-E(q_1)-E(q_2))\\ & \times \Big[F_T(p_1)F_T(p_2)\widetilde{F}_T(q_1)\widetilde{F}_T(q_2) - F_T(q_1)F_T(q_2)\widetilde{F}_T(p_1)\widetilde{F}_T(p_2)\Big],\end{aligned}$$

where $\widetilde{F}_T(p) := 1 - F_T(p)$. The derivation of (1.10) from the microscopic quantum dynamics is an extremely challenging open problem; for some work in this direction, see [4, 12, 16, 17, 22]. We note that (1.7) is also an equilibrium solution of (1.10), which is ealily seen by noting that $\widetilde{F}_0(p) = e^{\beta(E(p)-\mu)}F_0(p)$. This is a consequence of the circumstance that (1.7) is a function of the kinetic energy $E(p)$ which is a *collision invariant* in both (1.6) and (1.10). As a matter of fact, any distribution of the form $f(E(p))$ is stationary for (1.6); on the other hand, the special structure of (1.7) is necessary for it to be a stationary solution of (1.10). For a combined Boltzmann limit of the coupled model with $\lambda, \eta > 0$ (which is an open problem) we conjecture that the kinetic energy $E(p)$ will remain a collision invariant, and that the momentum distribution (1.7) will remain a stationary solution of the resulting Boltzmann equation, at least in a regime $\eta \le O(\lambda)$.

A contextually related question is the one addressing the stability of the Fermi sea for a gas of interacting fermions. This is an important problem in mathematical physics which has in recent years received much attention, especially due to the landmark works of Feldman, Knörrer, and Trubowitz summarized in [15].

An additional goal of the present work is to investigate the effective correlation between the electrons due to their interaction with the random potential. To this end, we assume that ρ_0 is number preserving, homogenous, and *quasifree*. That is, for any tuple of test functions $f_1, \dots, f_r, g_1, \dots, g_s$,

$$\rho_0(a^+(f_1)\cdots a^+(f_r)a(g_1)\cdots a(g_s)) = \delta_{r,s}\det[\rho_0(a^+(f_j)a(g_\ell))]_{j,\ell=1}^r. \tag{1.10}$$

We consider the dynamics generated by H_ω, and observe that since H_ω is bilinear in a^+, a, the time evolved state ρ_t is almost surely quasifree. However, the state averaged over the randomness is *not* quasifree,

$$\lim_{L\to\infty} \mathbb{E}[\rho_t(f_1,\dots,f_r;g_1,\dots,g_r)] \neq \det\Big[\lim_{L\to\infty} \mathbb{E}[\rho_t(f_j;g_\ell)]\Big]_{j,\ell=1}^r, \tag{1.11}$$

for any $t > 0$ if $\eta > 0$. This is not surprising because quasifreeness is a nonlinear condition. We prove that in the kinetic scaling limit stated above, the limiting $2r$-correlation functions are quasifree,

$$\begin{aligned}
&\Omega_T^{(2r)}(f_1,\dots,f_r;g_1,\dots,g_r)\\
&\quad := \lim_{\eta\to 0}\lim_{L\to\infty} \mathbb{E}[\rho_{T/\eta^2}(a^+(f_1)\cdots a^+(f_r)a(g_1)\cdots a(g_2))]\\
&\quad = \det[\Omega_T^{(2)}(f_j;g_\ell)]_{j,\ell=1}^r,
\end{aligned} \tag{1.12}$$

for any $r \in \mathbb{N}$. The proof is based on an extension of the proof in [8] for the 1-particle Anderson model at weak disorders that the random Schrödinger evolution converges in *arbitrary higher mean* to a linear Boltzmann evolution. Quasifreeness of the $2r$-point correlation functions is a significant ingredient in some approaches to the problem of quantum charge transport; see for instance [2] and the references therein.

2 Definition of the Model

We give a detailed definition of the mathematical model described in the previous section. We consider a fermion gas in a finite box $\Lambda_L := [-\frac{L}{2}, \frac{L}{2}]^d \cap \mathbb{Z}^d$ of side length $L \gg 1$, with periodic boundary conditions, in dimensions $d \geq 3$. We denote its dual lattice by $\Lambda_L^* := \Lambda_L/L \subset \mathbb{T}^d$. For the Fourier transform, we use the convention

$$\widehat{f}(p) := \sum_{x\in\Lambda_L} e^{-2\pi i p\cdot x} f(x), \tag{2.1}$$

where $p \in \Lambda_L^*$, and

$$f(x) = \frac{1}{L^d}\sum_{p\in\Lambda_L^*} e^{2\pi i p\cdot x}\widehat{f}(p) \tag{2.2}$$

for its inverse. For brevity, we will use the notation

$$\int dp \equiv \frac{1}{L^d}\sum_{p\in\Lambda_L^*} \tag{2.3}$$

in the sequel, which recovers its usual meaning in the thermodynamic limit $L \to \infty$.

We denote the fermionic Fock space of scalar electrons by

$$\mathfrak{F} = \bigoplus_{n\geq 0} \mathfrak{F}_n, \tag{2.4}$$

where

$$\mathfrak{F}_0 = \mathbb{C}, \qquad \mathfrak{F}_n = \bigwedge_1^n \ell^2(\Lambda_L), \quad n \geq 1. \tag{2.5}$$

We introduce creation- and annihilation operators a_p^+, a_q, for p, $q \in \Lambda_L^*$, satisfying the canonical anticommutation relations

$$a_p^+ a_q + a_q a_p^+ = \delta(p-q) := \begin{cases} L^d & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

We define the fermionic manybody Hamiltonian

$$H_\omega := T + \eta V_\omega, \tag{2.7}$$

where

$$T = \int dp E(p) a_p^+ a_p \tag{2.8}$$

is the kinetic energy operator, and

$$V_\omega := \sum_{x \in \Lambda_L} \omega_x a_x^+ a_x \tag{2.9}$$

couples the fermions to a static random potential; $\{\omega_x\}_{x \in \Lambda_L}$ is a field of i.i.d. real-valued random variables which we assume to be centered, normalized, and Gaussian for simplicity. Thus,

$$\mathbb{E}[\omega_x] = 0, \qquad \mathbb{E}[\omega_x^2] = 1 \tag{2.10}$$

for all $x \in \Lambda_L$. Moreover, we assume that

$$E(p) = \sum_{j=1}^d \cos(2\pi p_j), \tag{2.11}$$

which defines the Fourier multiplier corresponding to the nearest neighbor Laplacian on $\mathbb{Z}^d$.

Let

$$N := \sum_{x \in \Lambda_L} a_x^+ a_x \tag{2.12}$$

denote the particle number operator. It is clear that

$$[H_\omega, N] = 0 \tag{2.13}$$

holds.

Let $\mathfrak{A}$ denote the C^*-algebra of bounded operators on $\mathfrak{F}$. We consider the dynamics on $\mathfrak{A}$ given by

$$\alpha_t(A) = e^{itH_\omega} A e^{-itH_\omega} \tag{2.14}$$

generated by the random Hamiltonian H_ω.

3 Statement of the Main Results

We consider a normalized, translation-invariant, deterministic state

$$\rho_0 : \mathfrak{A} \longrightarrow \mathbb{C}. \tag{3.1}$$

We define the time-evolved state

$$\rho_t(A) := \rho_0(e^{itH_\omega} A e^{-itH_\omega}), \tag{3.2}$$

with $t \in \mathbb{R}$, and initial condition given by ρ_0. We particularly focus on the dynamics of the averaged two-point functions

$$\mathbb{E}[\rho_t(a_p^+ a_q)], \tag{3.3}$$

where $p, q \in \Lambda_L^*$. Clearly,

$$\mathbb{E}[\rho_0(a_p^+ a_q)] = \rho_0(a_p^+ a_q) = \delta(p-q)\frac{1}{L^d}\rho_0(a_p^+ a_p), \tag{3.4}$$

where

$$\delta(k) := L^d \delta_k, \tag{3.5}$$

and where

$$\delta_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

denotes the Kronecker delta on the lattice Λ_L^* (mod $\mathbb{T}^d$). We remark that for fermions,

$$0 \le \frac{1}{L^d}\rho_0(a_p^+ a_p) \le 1, \tag{3.7}$$

since $\|a_p^{(+)}\| = L^{d/2}$ in operator norm, $\forall p \in \Lambda_L^*$.

3.1 The Boltzmann Limit

We denote the microscopic time, position, and velocity variables by (t, x, p), and the corresponding macroscopic variables by $(T, X, V) = (\eta^2 t, \eta^2 x, v)$. We prove that the momentum distribution $f_t(q)$ converges to a solution of a linear Boltzmann equation in the limit $\eta \to 0$.

Theorem 3.1 *We assume that ρ_0 is translation invariant. Then, the averaged two-point functions are translation invariant,*

$$\mathbb{E}[\rho_t(a^+(f)a(g))] = \int dp\, \overline{f(p)} g(p) \mathbb{E}[\rho_t(a_p^+ a_p)] \tag{3.8}$$

(i.e., diagonal in a_p^+, a_p) for any $f, g \in \mathcal{S}(\mathbb{T}^d)$ of Schwartz class, and the thermodynamic limit

$$\Omega_T^{(2;\eta)}(f; g) := \lim_{L\to\infty} \mathbb{E}[\rho_{T/\eta^2}(a^+(f)a(g))] \tag{3.9}$$

exists for all $f, g \in \mathcal{S}(\mathbb{T}^d)$, *and* $T > 0$.

For any $T > 0$ *and all* $f, g \in \mathcal{S}(\mathbb{T}^d)$, *the limit*

$$\Omega_T^{(2)}(f; g) := \lim_{\eta \to 0} \Omega_T^{(2;\eta)}(f; g) \tag{3.10}$$

exists, and is the inner product of f, g *with respect to a Borel measure* $F_T(p)dp$,

$$\Omega_T^{(2)}(f; g) = \int dp F_T(p) \overline{f(p)} g(p), \tag{3.11}$$

where $F_T(V)$ *satisfies the linear Boltzmann equation*

$$\partial_T F_T(V) = 2\pi \int_{\mathbb{T}^d} dU \delta(E(U) - E(V))(F_T(U) - F_T(V)), \tag{3.12}$$

with initial condition

$$F_0(p) = \lim_{L \to \infty} \frac{1}{L^d} \rho_0(a^+_{p_{\Lambda_L^*}} a_{p_{\Lambda_L^*}}) \tag{3.13}$$

for $p \in \mathbb{T}^d$, *where* $p_{\Lambda_L^*} := Q_{\frac{1}{2L}}(p) \cap \Lambda_L^*$, *and* $Q_\delta(p) := p + [-\delta, \delta)^d$.

We note that there exists a unique $p_{\Lambda_L^*} \in \Lambda_L^*$ such that $|p - p_{\Lambda_L^*}| \le \frac{1}{2L}$, for every $p \in \mathbb{T}^d$.

An initial condition of particular interest is the Gibbs state (with inverse temperature β and chemical potential μ) for a non-interacting fermion gas,

$$\rho_0(A) = \frac{1}{Z_{\beta,\mu}} \mathrm{Tr}(e^{-\beta(T-\mu N)} A) \tag{3.14}$$

where $Z_{\beta,\mu} := \mathrm{Tr}(e^{-\beta(T-\mu N)})$. The corresponding momentum distribution function

$$\lim_{L \to \infty} \frac{1}{L^d} \rho_0(a_p^+ a_p) = \frac{1}{1 + e^{\beta(E(p)-\mu)}} \tag{3.15}$$

is a *stationary solution* of the linear Boltzmann equation (3.12), for all $\beta > 0$. This also holds in the zero temperature limit $\beta \to \infty$ where in the weak sense,

$$\frac{1}{1 + e^{\beta(E(p)-\mu)}} \to \chi[E(p) < \mu], \tag{3.16}$$

which is nontrivial if $\mu > 0$. We note that all our results in this paper remain valid in the limit $\beta \to \infty$.

3.2 Quasifreeness

We prove that if in addition to the conditions formulated above, the initial state ρ_0 is *quasifree*, then $\mathbb{E}[\rho_t]$, which is not quasifree for any $t > 0$ if $\eta > 0$, becomes quasifree in the kinetic scaling limit of Theorem 3.1.

A state ρ_0 is quasifree if for any normal ordered product of creation- and annihilation operators

$$a^+_{p_1} \cdots a^+_{p_r} a_{q_1} \cdots a_{q_s}, \tag{3.17}$$

with arbitrary $r, s \in \mathbb{N}$ and $p_i, q_j \in \Lambda_L^*$,

$$\rho_0(a_{p_1}^+ \cdots a_{p_r}^+ a_{q_1} \cdots a_{q_s}) = \delta_{r,s} \det\big[\rho_0(a_{p_i}^+ a_{q_j})\big]_{1\le i,j\le r}. \tag{3.18}$$

That is, any higher order correlation function decomposes into the determinant of the matrix of pair correlations. In its most general form, a particle number conserving quasifree state $\rho_0 : \mathfrak{A} \to \mathbb{C}$ can be written as

$$\rho_0(A) := \frac{1}{Z_K} \mathrm{Tr}(e^{-K} A) \tag{3.19}$$

for $A \in \mathfrak{A}$, with

$$Z_K := \mathrm{Tr}(e^{-K}), \tag{3.20}$$

and

$$K = \int dp dq \kappa(p,q) a_p^+ a_q \tag{3.21}$$

bilinear in a_p^+, a_q; for a proof, see [3]. We assume K to be deterministic (with respect to $\{\omega_x\}_x$).

If in addition, translation invariance is imposed, such that

$$[K, T] = 0 \tag{3.22}$$

then

$$K = \int dp h(p) a_p^+ a_p \tag{3.23}$$

is bilinear and diagonal in a_p^+, a_p.

Since H_ω is bilinear in the creation- and annihilation operators, it is immediately clear that

$$K(t) := e^{itH_\omega} K e^{-itH_\omega} \tag{3.24}$$

is also bilinear in a_p^+, a_q. Therefore,

$$\rho_t(A) = \frac{1}{Z_K} \mathrm{Tr}(e^{-K(t)} A) \tag{3.25}$$

is quasifree with probability 1. However, since quasifreeness is a *nonlinear* condition on determinants, almost sure quasifreeness does *not* imply that $\mathbb{E}[\rho_t(\cdot)]$ is quasifree.

In fact, $\mathbb{E}[\rho_t(\cdot)]$ is *not* quasifree for any $t > 0$.

However, we prove in Theorem 3.2 below that it possesses a kinetic scaling limit (in the sense of Theorem 3.1) which is quasifree.

Theorem 3.2 *Assume that ρ_0 is number conserving and quasifree, and translation invariant. Then, the following holds. For any normal ordered monomial in creation- and annihilation operators,*

$$a^+(f_1) \cdots a^+(f_r) a(g_1) \cdots a(g_r), \tag{3.26}$$

with $r, s \in \mathbb{N}$ and Schwartz class test functions $f_j, g_\ell \in \mathcal{S}(\mathbb{T}^d)$, and any $T > 0$, the macroscopic $2r$-point function

$$\begin{aligned}\Omega_T^{(2r)}&(f_1, \dots, f_r; g_1, \dots, g_r)\\ &:= \lim_{\eta\to 0}\lim_{L\to\infty} \mathbb{E}[\rho_{T/\eta^2}(a^+(f_1)\cdots a^+(f_r)a(g_1)\cdots a(g_r))]\end{aligned} \tag{3.27}$$

exists and is quasifree,

$$\Omega_T^{(2r)}(f_1, \dots, f_r; g_1, \dots, g_r) = \det\big[\Omega_T^{(2)}(f_i, g_j)\big]_{1\le i,j\le r}. \tag{3.28}$$

The macroscopic 2-point function is the same as in Theorem 3.1,

$$\Omega_T^{(2)}(f; g) = \int dp F_T(p)\overline{f(p)}g(p), \tag{3.29}$$

and $F_T(p)$ solves the linear Boltzmann equation (3.12) with initial condition (3.13).

We note that the assumption of translation invariance can easily be dropped. However, we do not address inhomogenous Fermi gases in this text.

4 Proof of Theorem 3.1

The proof of Theorem 3.1 is obtained from an extension of the analysis in [7, 11].

4.1 Duhamel Expansion

We consider the Heisenberg evolution of the creation- and annihilation operators. We define

$$a_p(t) := e^{itH_\omega}a_p e^{-itH_\omega}, \tag{4.1}$$

and

$$a(f, t) := e^{itH_\omega}a(f)e^{-itH_\omega}. \tag{4.2}$$

We make the key observation that

$$a(f, t) = a(f_t) \tag{4.3}$$

where f_t is the solution of the 1-particle random Schrödinger equation

$$i\partial_t f_t = H_\omega^{(1)} f_t := \Delta f_t + \eta V_\omega^{(1)} f_t \tag{4.4}$$

with initial condition

$$f_0 = f. \tag{4.5}$$

Here, Δ denotes the nearest neighbor Laplacian on Λ_L, and $H_\omega^{(1)} = H_\omega|_{\mathfrak{F}_1}$ is the 1-particle Anderson Hamiltonian at weak disorders studied in [7, 8, 11]. $V_\omega^{(1)} = V_\omega|_{\mathfrak{F}_1}$ is the 1-particle multiplication operator $(V_\omega^{(1)} f)(x) = \omega_x f(x)$.

To prove (4.4), (4.5), we observe that since H_ω is bilinear in a^+, a, it follows that $a(f,t)$ is a linear superposition of annihilation operators. Therefore, there exists a function f_t such that $a(f,t)=a(f_t)$. In particular,

$$\begin{aligned} i\partial_t a(f_t) &= [H_\omega, a(f_t)] \\ &= \int dp f_t(p)E(p)a_p + \eta\int dp\int du f_t(p)\widehat{\omega}(u-p)a_u \\ &= a(\Delta f_t) + a(\eta V_\omega^{(1)} f_t), \end{aligned} \tag{4.6}$$

and moreover, it is clear that $a(f,0)=a(f_0)=a(f)$. This implies (4.4), (4.5).

Thus,

$$\begin{aligned} \rho_t(a^+(f)a(g)) &= \rho_0(a^+(f_t)a(g_t)) \\ &= \int dpdq\rho_0(a_p^+a_q)\overline{f_t(p)}g_t(q) \\ &= \int dpJ(p)\overline{f_t(p)}g_t(p), \end{aligned} \tag{4.7}$$

where

$$\rho_0(a_p^+a_q) = \delta(p-q)J(p) \tag{4.8}$$

due to translation invariance, with

$$0\le J(p) = \frac{1}{L^d}\rho_0(a_p^+a_p) = \frac{1}{1+e^{h(p)}} \le 1, \tag{4.9}$$

cf. (3.7); see (3.23) for the definition of $h(p)$. In particular, this implies (3.8).

For $N\in\mathbb{N}$, which we determine later, we expand f_t, g_t into the truncated Duhamel series at level N,

$$f_t = f_t^{(\le N)} + f_t^{(>N)}, \tag{4.10}$$

with

$$f_t^{(\le N)} := \sum_{n=0}^{N} f_t^{(n)}, \tag{4.11}$$

and where the Duhamel term of n-th order (in powers of η) is given by

$$\begin{aligned} f_t^{(n)}(p) &:= (i\eta)^n \int ds_0\cdots ds_n\delta\Bigg(t-\sum_{j=0}^n s_j\Bigg) \\ &\quad\times\int dk_0\cdots dk_n\delta(p-k_0)\Bigg(\prod_{j=0}^n e^{is_jE(k_j)}\Bigg)\Bigg(\prod_{j=1}^n\widehat{\omega}(k_j-k_{j-1})\Bigg)f(k_n) \end{aligned} \tag{4.12}$$

$$\begin{aligned} &= \eta^n e^{\epsilon t}\int d\alpha e^{it\alpha}\int dk_0\cdots dk_n\delta(p-k_0) \\ &\quad\times\Bigg(\prod_{j=0}^n\frac{1}{E(k_j)-\alpha-i\epsilon}\Bigg)\Bigg(\prod_{j=1}^n\widehat{\omega}(k_j-k_{j-1})\Bigg)f(k_n). \end{aligned} \tag{4.13}$$

The remainder term is given by

$$f_t^{(>N)} = i\eta \int_0^t ds\, e^{i(t-s)H_\omega} V_\omega^{(1)} f_t^{(N)}(s). \tag{4.14}$$

We choose

$$\epsilon = \frac{1}{t} \tag{4.15}$$

so that the factor $e^{\epsilon t}$ remains bounded for all t. Accordingly,

$$\rho_t(a^+(f)a(g)) = \rho_0(a^+(f_t)a(g_t)) = \sum_{n,\widetilde{n}=0}^{N+1} \rho_t^{(n,\widetilde{n})}(f;g), \tag{4.16}$$

where

$$\rho_t^{(n,\widetilde{n})}(f;g) := \rho_0(a^+(f_t^{(n)})a(g_t^{(\widetilde{n})})) \tag{4.17}$$

if $n, \widetilde{n} \le N$, and

$$\begin{aligned} \rho_t^{(n,N+1)}(f;g) &:= \rho_0(a^+(f_t^{(n)})a(g_t^{(>N)})), \\ \rho_t^{(N+1,\widetilde{n})}(f;g) &:= \rho_0(a^+(f_t^{(>N)})a(g_t^{(\widetilde{n})})) \end{aligned} \tag{4.18}$$

if $n \le N$, respectively if $\widetilde{n} \le N$, and

$$\rho_t^{(N+1,N+1)}(f;g) := \rho_0(a^+(f_t^{(>N)})a(g_t^{(>N)})). \tag{4.19}$$

In particular, for $n, \widetilde{n} \le N$,

$$\begin{aligned} \rho_t^{(n,\widetilde{n})}(f;g) = \eta^{n+\widetilde{n}} e^{2\epsilon t} \int d\alpha\, d\widetilde{\alpha}\, e^{it(\alpha-\widetilde{\alpha})} \\ \times \int dk_0 \cdots dk_n \int d\widetilde{k}_0 \cdots d\widetilde{k}_{\widetilde{n}} \overline{f(k_n)} g(\widetilde{k}_{\widetilde{n}}) J(k_0) \delta(k_0 - \widetilde{k}_0) \\ \times \prod_{j=0}^{n} \frac{1}{E(k_j) - \alpha - i\epsilon} \prod_{\ell=0}^{n} \frac{1}{E(\widetilde{k}_\ell) - \widetilde{\alpha} + i\epsilon} \\ \times \prod_{j=1}^{n} \widehat{\omega}(k_j - k_{j-1}) \prod_{\ell=1}^{n} \widehat{\omega}(\widetilde{k}_{\ell-1} - \widetilde{k}_\ell). \end{aligned} \tag{4.20}$$

This expression, and the expressions involving n and / or $\widetilde{n} = N+1$, are completely analogous to those appearing in the truncated Duhamel expansion of the Wigner transform in [7, 11].

This permits us to use the methods of [7, 11] to prove Theorem 3.1. We will here only sketch the strategy; for the detailed proof, we refer to [7, 11]. In our subsequent discussion, we will compare the expressions appearing in the given problem to those treated in [7, 11].

To begin with, we introduce a more convenient notation. Clearly, if $n, \widetilde{n} \le N$, and $n + \widetilde{n}$ is odd, $\mathbb{E}[\rho_t^{(n,\widetilde{n})}(p,q)] = 0$. Thus, we let

$$\bar{n} := \frac{n+\widetilde{n}}{2} \in \mathbb{N}, \tag{4.21}$$

and we define $\{u_j\}_{j=0}^{2\bar{n}+1}$ by

$$u_j := \begin{cases} k_{n-j} & \text{if } j \leq n \\ \widetilde{k}_{j-n-1} & \text{if } j \geq n+1. \end{cases} \tag{4.22}$$

Consequently,

$$\begin{aligned} \mathbb{E}[\rho_t^{(n,\widetilde{n})}(f;g)] = \eta^{2\bar{n}} e^{2\epsilon t} \int d\alpha d\widetilde{\alpha} e^{it(\alpha-\widetilde{\alpha})} \\ \times \int du_0 \cdots du_{2\bar{n}+1} \overline{f(u_0)} g(u_{2\bar{n}+1}) J(u_n) \delta(u_n - u_{n+1}) \\ \times \prod_{j=0}^{n} \frac{1}{E(u_j) - \alpha - i\epsilon} \prod_{\ell=n+1}^{2\bar{n}+1} \frac{1}{E(u_\ell) - \widetilde{\alpha} + i\epsilon} \\ \times \mathbb{E}\Bigg[\prod_{j=1}^{n} \widehat{\omega}(u_j - u_{j-1}) \prod_{j=n+2}^{2\bar{n}+1} \widehat{\omega}(u_j - u_{j-1}) \Bigg] \end{aligned} \tag{4.23}$$

in these new variables, where we use that $\widehat{\omega}(u)^* = \widehat{\omega}(-u)$.

4.2 Graph Expansion

Next, we take the expectation with respect to the random potential. To this end, we introduce the set of *Feynman graphs* $\Gamma_{n,\widetilde{n}}$, with $n + \widetilde{n} \in 2\mathbb{N}$, as follows.

We consider two horizontal solid lines, which we refer to as *particle lines*, joined by a distinguished vertex which we refer to as the ρ_0-vertex (corresponding to the term $\rho_0(a_{u_n}^+ a_{u_{n+1}})$). On the line on its left, we introduce n vertices, and on the line on its right, we insert $\widetilde{n}$ vertices. We refer to those vertices as *interaction vertices*, and enumerate them from 1 to $2\bar{n}$ starting from the left. The edges between the interaction vertices are referred to as *propagator lines*. We label them by the momentum variables $u_0, \ldots, u_{2\bar{n}+1}$, increasingly indexed starting from the left. To the j-th propagator line, we associate the resolvent $\frac{1}{E(u_j)-\alpha-i\epsilon}$ if $0 \leq j \leq n$, and $\frac{1}{E(u_j)-\widetilde{\alpha}+i\epsilon}$ if $n+1 \leq j \leq 2\bar{n}+1$. To the ℓ-th interaction vertex (adjacent to the edges labeled by $u_{\ell-1}$ and u_ℓ), we associate the random potential $\widehat{\omega}(u_\ell - u_{\ell-1})$, where $1 \leq \ell \leq 2\bar{n}+1$.

A *contraction graph* associated to the above pair of particle lines joined by the ρ_0-vertex, and decorated by $n + \widetilde{n}$ interaction vertices, is the graph obtained by pairwise connecting interaction vertices by dashed *contraction lines*. We denote the set of all such contraction graphs by $\Gamma_{n,\widetilde{n}}$; it contains

$$|\Gamma_{n,\widetilde{n}}| = (2\bar{n}-1)(2\bar{n}-3)\cdots 3\cdot 1 = \frac{(2\bar{n})!}{\bar{n}!2^{\bar{n}}} = O(\bar{n}!) \tag{4.24}$$

elements.

If in a given graph $\pi \in \Gamma_{n,\widetilde{n}}$, the ℓ-th and the ℓ'-th vertex are joined by a contraction line, we write

$$\ell \sim_\pi \ell', \tag{4.25}$$

and we associate the delta distribution

$$\delta(u_\ell - u_{\ell-1} - (u_{\ell'} - u_{\ell'-1})) = \mathbb{E}[\widehat{\omega}(u_\ell - u_{\ell-1})\widehat{\omega}(u_{\ell'} - u_{\ell'-1})] \tag{4.26}$$

to this contraction line.

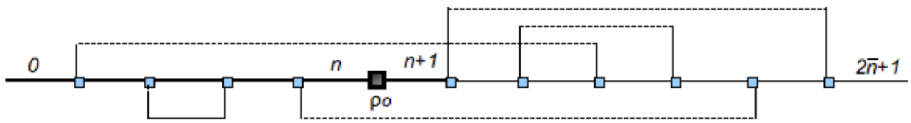

Fig. 1 An example of a Feynman graph, $\pi \in \Gamma_{n,\widetilde{n}}$, with $n=4$, $\widetilde{n}=6$. The distinguished vertex is the ρ_0-vertex

4.3 Classification of Graphs

For the proof of Theorem 3.1, we classify Feynman graphs as follows; see [7, 11], and Fig. 1.

- A subgraph consisting of one propagator line adjacent to a pair of vertices ℓ and $\ell+1$, and a contraction line connecting them, i.e., $\ell \sim_\pi \ell+1$, where both ℓ, $\ell+1$ are either $\leq n$ or $\geq n+1$, is called an *immediate recollision*.
- The graph $\pi \in \Gamma_{n,n}$ (i.e., $n=\widetilde{n}=\bar{n}$) with $\ell \sim_\pi 2n-\ell$ for all $\ell = 1,\dots,n$, is called a *basic ladder* diagram. The contraction lines are called *rungs* of the ladder. We note that a rung contraction always has the form $\ell \sim_\pi \ell'$ with $\ell \leq n$ and $\ell' \geq n+1$. Moreover, in a basic ladder diagram one always has that if $\ell_1 \sim_\pi \ell_1'$ and $\ell_2 \sim_\pi \ell_2'$ with $\ell_1 < \ell_2$, then $\ell_2' < \ell_1'$.
- A diagram $\pi \in \Gamma_{n,\widetilde{n}}$ is called a *decorated ladder* if any contraction is either an immediate recollision, or a rung contraction $\ell_j \sim_\pi \ell_j'$ with $\ell_j \leq n$ and $\ell_j' \geq n$ for $j = 1,\dots,k$, and $\ell_1 < \cdots < \ell_k$, $\ell_1' > \cdots > \ell_k'$. Evidently, a basic ladder diagram is the special case of a decorated ladder which contains no immediate recollisions (so that necessarily, $n=\widetilde{n}$).
- A diagram $\pi \in \Gamma_{n,\widetilde{n}}$ is called *crossing* if there is a pair of contractions $\ell \sim_\pi \ell'$, $j \sim_\pi j'$, with $\ell < \ell'$ and $j < j'$, such that $\ell < j$.
- A diagram $\pi \in \Gamma_{n,\widetilde{n}}$ is called *nesting* if there is a subdiagram with $\ell \sim_\pi \ell+2k$, with $k \geq 1$, and either $\ell \geq n+1$ or $\ell+2k \leq n$, with $j \sim_\pi j+1$ for $j = \ell+1, \ell+3, \dots, \ell+2k-1$. The latter corresponds to a progression of $k-1$ immediate recollisions.

We note that any diagram that is not a decorated ladder contains at least a crossing or a nesting subdiagram.

4.4 Feynman Amplitudes

Next, we average (4.20) with respect to the random potential. Accordingly, $\mathbb{E}[\prod \widehat{\omega}(u_\ell - u_{\ell-1})]$ splits into the sum of all possible products of pair correlations, according to Wick's theorem (we recall that $\{\omega_x\}$ are assumed to be i.i.d. Gaussian). This implies that

$$\mathbb{E}[\rho_t^{(n,\widetilde{n})}(f;g)] = \sum_{\pi \in \Gamma_{n,\widetilde{n}}} \mathrm{Amp}_\pi(f;g;\epsilon;\eta) \tag{4.27}$$

with

$$\begin{aligned} \mathrm{Amp}_\pi(f;g;\epsilon;\eta) &:= \eta^{2\bar{n}} e^{2\epsilon t} \int d\alpha d\widetilde{\alpha} e^{it(\alpha-\widetilde{\alpha})} \\ &\quad \times \int du_0 \cdots du_{2\bar{n}+1} \overline{f(u_0)} g(u_{2\bar{n}+1}) J(u_n) \delta(u_n - u_{n+1}) \delta_\pi(\{u_j\}_{j=0}^{2\bar{n}+1}) \\ &\quad \times \prod_{j=0}^{n} \frac{1}{E(u_j)-\alpha-i\epsilon} \prod_{\ell=n+2}^{2\bar{n}} \frac{1}{E(u_\ell)-\widetilde{\alpha}+i\epsilon}, \end{aligned} \tag{4.28}$$

and $\epsilon = \frac{1}{t}$. Here,

$$\delta_\pi(\{u_j\}_{j=0}^{2\bar{n}+1}) := \prod_{\ell \sim_\pi \ell'} \delta(u_\ell - u_{\ell-1} - (u_{\ell'} - u_{\ell'-1})) \tag{4.29}$$

is the product of the delta distributions associated to all contraction lines in π. Moreover, we recall that

$$\delta(u_n - u_{n+1}) J(u_n) = \rho_0(a_{u_n}^+ a_{u_{n+1}}), \tag{4.30}$$

see (4.8). We note that

$$u_0 - u_{2\bar{n}+1} = 0, \tag{4.31}$$

as one easily sees by summing up the arguments of all delta distributions. This holds for any $n, \widetilde{n}$ and again implies (3.8).

We observe that the rôle of (4.30) in (4.28) is analogous to that of the rescaled Schwartz class function J_ϵ in [7, 11], and that the test functions f, g here correspond to the initial state $\widehat{\phi}_0$ in [7, 11].

4.5 Contribution from Crossing and Nesting Diagrams

The amplitude of any graph $\pi \in \Gamma_{n,\widetilde{n}}$ that contains either a crossing or a nesting can be estimated by

$$\lim_{L\to\infty} |\mathrm{Amp}_\pi(f; g; \epsilon; \eta)| \le \|f\|_2 \|g\|_2 \|J\|_\infty \epsilon^{1/5} \left(\log \frac{1}{\epsilon}\right)^4 \left(c\eta^2 \epsilon^{-1} \log \frac{1}{\epsilon}\right)^{\bar{n}}, \tag{4.32}$$

see [7, 11]. We note that similarly as in [7, 11], the bounds on all error terms will only depend on the L^2-norm of the initial condition, which in [7, 11] is normalized by $\|\widehat{\phi}_0\|_2^2 = 1$.

The existence of the thermodynamic limit, as $L \to \infty$, is obtained precisely in the same manner as in [7, 8]. Let

$$\Gamma_{n,\widetilde{n}}^{c-n} \subset \Gamma_{n,\widetilde{n}} \tag{4.33}$$

denote the subset of diagrams of crossing or nesting type. The number of graphs in

$$\Gamma_{2\bar{n}}^{c-n} := \bigcup_{n+\widetilde{n}=2\bar{n}} \Gamma_{n,\widetilde{n}}^{c-n} \tag{4.34}$$

is bounded by $2^{\bar{n}} \bar{n}!$.

Thus, the sum of amplitudes associated to all crossing and nesting diagrams is bounded by

$$\sum_{1\le\bar{n}\le N} \sum_{\pi\in\Gamma_{2\bar{n}}^{c-n}} \lim_{L\to\infty} |\mathrm{Amp}_\pi(f; g; \epsilon; \eta)|$$

$$< (N+1)!\epsilon^{1/5} \left(\log \frac{1}{\epsilon}\right)^4 \left(c\eta^2 \epsilon^{-1} \log \frac{1}{\epsilon}\right)^N \tag{4.35}$$

noting that evidently, $\|f\|_2, \|g\|_2 < C$ for f, g of Schwartz class, and recalling from (3.7) that

$$\|J\|_\infty \leq 1, \tag{4.36}$$

which in particular is the case for $J(p) = (1 + e^{\beta(E(p)-\mu)})^{-1}$ associated to a Gibbs state of the free Fermi field, for all $0 \leq \beta \leq \infty$.

4.6 Remainder Term and Time Partitioning

If at least one of the indices $n, \widetilde{n}$ equals $N+1$, we first use

$$|\mathbb{E}[\rho_t^{(N+1,\widetilde{n})}(f; g)]| \leq (\mathbb{E}[\rho_t^{(\widetilde{n},\widetilde{n})}(g; g)])^{1/2} (\mathbb{E}[\rho_t^{(N+1,N+1)}(f; f)])^{1/2} \tag{4.37}$$

by the Schwarz inequality (assuming without any loss of generality that $n = N+1$). If $\widetilde{n} \leq N$, the term $\mathbb{E}[\rho_t^{(\widetilde{n},\widetilde{n})}(g; g)]$ admits a bound of the form (4.47) below.

To bound $\mathbb{E}[\rho_t^{(N+1,N+1)}(f; f)]$, corresponding to the remainder term in the Duhamel expansion, we use the time partitioning method of [11]; see also [7]. To this end, we further expand the remainder term into $3N$ additional Duhamel terms, and to subdivide the time integration interval $[0, t]$ into $\kappa \in \mathbb{N}$ equal segments

$$[0, t] = \bigcup_{j=1}^{\kappa} [\tau_{j-1}, \tau_j], \qquad \tau_j = \frac{jt}{\kappa}, \tag{4.38}$$

whereby one obtains

$$f_t^{(>N)} = f_t^{(N,4N)} + f_t^{(>4N)}, \tag{4.39}$$

where

$$f_t^{(N,4N)} := \sum_{j=1}^{\kappa} \sum_{n=N+1}^{4N-1} e^{i(t-\tau_j)H_\omega^{(1)}} \widetilde{f}_{\tau_j}^{(n,N,\tau_{j-1})}, \tag{4.40}$$

with

$$\widetilde{f}_s^{(n,N,\tau_{j-1})} := V_\omega^{(1)} f_s^{(n,N,\tau_{j-1})}, \tag{4.41}$$

and

$$\begin{aligned} f_s^{(n,N,\tau)}(p) &:= (i\eta)^{n-N} \int_{\mathbb{R}^{n-N+1}} ds_0 \cdots ds_{n-N} \delta\left(\sum_{j=0}^{n-N} s_j - (s-\tau)\right) \\ &\quad \times \int du_0 \cdots du_{n-N} \delta(p - u_0) \prod_{j=0}^{n-N} e^{is_j E(u_j)} \\ &\quad \times \prod_{\ell=1}^{n-N} \widehat{\omega}(u_j - u_{j-1}) f(u_{n-N}). \end{aligned} \tag{4.42}$$

Moreover,

$$f_t^{(>4N)} = \sum_{j=1}^{\kappa} e^{i(t-\tau_j)H_\omega} \int_{\tau_{j-1}}^{\tau_j} ds e^{i(\tau_j - s)H_\omega^{(1)}} \widetilde{f}_s^{(N,4N,\tau_{j-1})}.$$

We note that writing (4.40) in the form $\sum_{j=1}^{\kappa}\sum_{n=N+1}^{4N-1} g_{n,j}$, we have

$$\begin{aligned}
&\rho_t(a^+(f_t^{(N,4N)})a(f_t^{(N,4N)})) \\
&\quad\le \sum_{j,j'=1}^{\kappa}\sum_{n,n'=N+1}^{4N-1} \left|\rho_0(a^+(g_{n',j'})a(g_{n,j}))\right| \\
&\quad\le \sum_{j,j'=1}^{\kappa}\sum_{n,n'=N+1}^{4N-1} \frac{1}{2}\Big[\rho_0(a^+(g_{n',j'})a(g_{n',j'})) + \rho_0(a^+(g_{n,j})a(g_{n,j}))\Big] \\
&\quad\le \kappa^2(3N)^2 \sup_{n,j} \rho_0(a^+(g_{n,j})a(g_{n,j})).
\end{aligned} \tag{4.43}$$

Thus, by the Schwarz inequality,

$$\rho_t^{(N+1,N+1)}(f;f) \le 2\Big[R_1(f,t) + R_2(f,t)\Big], \tag{4.44}$$

where

$$R_1(f,t) := (3N)^2\kappa^2 \sup_{\substack{N<n<4N\\ 1\le j\le\kappa}} \rho_0(a^+(f_{\tau_j}^{(n,N,\tau_{j-1})})a(f_{\tau_j}^{(n,N,\tau_{j-1})})) \tag{4.45}$$

and

$$R_2(f,t) := t^2 \sup_{1\le j\le\kappa}\ \sup_{s\in[\tau_{j-1},\tau_j]} \rho_0(a^+(\widetilde{f}_s^{(N,4N,\tau_{j-1})})a(\widetilde{f}_s^{(N,4N,\tau_{j-1})})). \tag{4.46}$$

By separating terms due to decorated ladders from those due to crossing and nesting diagrams, one finds

$$\begin{aligned}
&\lim_{L\to\infty} \mathbb{E}[\rho_0(a^+(f_{\tau_j}^{(n,N,\tau_{j-1})})a(f_{\tau_j}^{(n,N,\tau_{j-1})}))] \\
&\quad= \mathbb{E}\Big[\int dp J(p)|f_{\tau_j}^{(n,N,\tau_{j-1})}(p)|^2\Big] \\
&\quad\le \|J\|_\infty \mathbb{E}[\|f_{\tau_j}^{(n,N,\tau_{j-1})}\|_2^2] \\
&\quad\le \|f\|_2^2\|J\|_\infty\left[\frac{(c\epsilon^{-1}\eta^2)}{(N!)^{1/2}} + \epsilon^{1/5}\left(\log\frac{1}{\epsilon}\right)^4\left(c\eta^2\epsilon^{-1}\log\frac{1}{\epsilon}\right)^{8N}\right]
\end{aligned} \tag{4.47}$$

for $N<n<4N$ (see [7, 11] for a detailed discussion).

For $n=4N$, the main issue is to control the large factor t^2 in (4.46). To this end, we observe that for a time integral on the interval $[\tau_{j-1},\tau_j]$ of length $\frac{t}{\kappa}$, the parameter $\epsilon = t^{-1}$ can be replaced by $\kappa\epsilon = (\frac{t}{\kappa})^{-1}$. Therefore, one gets

$$\lim_{L\to\infty} \mathbb{E}[\rho_0(a^+(\widetilde{f}_s^{(N,4N,\tau_{j-1})})a(\widetilde{f}_s^{(N,4N,\tau_{j-1})}))]$$

$$\leq \|J\|_\infty \mathbb{E}[\|\widetilde{\widehat{f}_s}^{(N,4N,\tau_{j-1})}\|_2^2]$$
$$\leq \|f\|_2^2 \|J\|_\infty \left[\frac{((4N)!)}{\kappa^{2N}} \left(\log \frac{1}{\epsilon}\right)^4 \left(c\eta^2 \epsilon^{-1} \log \frac{1}{\epsilon}\right)^{8N} \right]. \tag{4.48}$$

The gain of a factor κ^{-2N} is crucial; it is sufficient to compensate for the factor t^2 in (4.46), using the parameter choice given in Sect. 4.7 below.

One obtains that if at least one of the indices $n, \widetilde{n}$ equals $N+1$,

$$\lim_{L\to\infty} |\mathbb{E}[\rho^{(n,\widetilde{n})}(f;f)]|$$
$$\leq \|f\|_2^2 \|J\|_\infty$$
$$\times \left[\frac{N^2\kappa^2(c\epsilon^{-1}\lambda^2)}{(N!)^{1/2}} + \left(N^2\kappa^2\epsilon^{1/5} + \epsilon^{-2}\kappa^{-2N}\right)((4N)!)\left(\log\frac{1}{\epsilon}\right)^4 \left(c\lambda^2\epsilon^{-1}\log\frac{1}{\epsilon}\right)^{8N}\right], \tag{4.49}$$

where κ remains to be chosen. The first term on the right hand side of (4.49) bounds the contribution from all basic ladder diagrams contained in the Duhamel expanded remainder term. For a detailed discussion, we refer to [7, 8, 11].

4.7 Choosing the Constants

We recall from (4.36) that $\|J\|_\infty \leq 1$. Moreover, $\|f\|_2$, $\|g\|_2 < C$ for all test functions f, $g \in \mathcal{S}(\mathbb{T}^d)$. As in [7, 8, 11], we choose

$$t = \frac{1}{\epsilon} = \frac{T}{\eta^2},$$
$$N = \frac{\log\frac{1}{\epsilon}}{10\log\log\frac{1}{\epsilon}}, \tag{4.50}$$
$$\kappa = \left(\log\frac{1}{\epsilon}\right)^{15}.$$

Then,

$$\epsilon^{-1/11} < N! < \epsilon^{-1/10},$$
$$\kappa^N > \epsilon^{-3/2} \tag{4.51}$$

and consequently,

$$(4.35), (4.48) < \eta^{1/15} \tag{4.52}$$

and

$$(4.49) < \eta^{1/4} \tag{4.53}$$

for η sufficiently small. It follows that the sum of all crossing, nesting, and remainder terms is bounded by $\eta^{1/20}$.

4.8 Resummation of Decorated Ladder Diagrams

Let $\Gamma^{(lad)}_{n,\widetilde{n}} \subset \Gamma_{n,\widetilde{n}}$ denote the subset of all decorated ladders based on $n+\widetilde{n}$ vertices. Then, for $T>0$, let

$$\Omega_T^{(2;\eta)}(f;g) := \sum_{\bar{n}=0}^{N(\epsilon(T,\eta))} \sum_{n+\widetilde{n}=2\bar{n}} \sum_{\pi\subset\Gamma^{(lad)}_{n,\widetilde{n}}} \lim_{L\to\infty} \mathrm{Amp}_\pi(f;g;\epsilon(T,\eta);\eta) \tag{4.54}$$

with $\epsilon(T,\eta)=\frac{\eta^2}{T}$. In the kinetic scaling limit $\eta\to 0$ with $t=\frac{1}{\epsilon}=T/\eta^2$, one obtains

$$\Omega_T^{(2)}(f;g) := \lim_{\eta\to 0}\Omega_T^{(2;\eta)}(f;g) = \int dp\, F_T(p)\overline{f(p)}g(p), \tag{4.55}$$

where

$$\begin{aligned} F_T(p) &:= \lim_{\eta\to 0} F_T^{(\eta)}(p) \\ &= e^{-2\pi T\int du (E(u)-E(p))} \sum_{\bar{n}=0}^{\infty}\int_{\mathbb{R}_+^{\bar{n}+1}} dS_0\cdots dS_{\bar{n}}\,\delta\left(T-\sum_{j=0}^{\bar{n}} S_j\right) \\ &\quad\times \int du_0\cdots du_n\,\delta(p-u_0)\left(\prod_{j=1}^{\bar{n}} 2\pi\delta(E(u_j)-E(u_{j-1}))\right)F_0(u_n), \end{aligned} \tag{4.56}$$

with initial condition

$$F_0(u) = \lim_{L\to\infty} J(u_{\Lambda_L^*}) = \lim_{L\to\infty}\frac{1}{L^d}\rho_0(a^+_{u_{\Lambda_L^*}} a_{u_{\Lambda_L^*}}) \tag{4.57}$$

(for the definition of $u_{\Lambda_L^*}$, see Theorem 3.1). It can be straightforwardly verified that (4.56) is a solution of the Cauchy problem for the linear Boltzmann equation (3.12), as asserted in Theorem 3.1.

5 Proof of Theorem 3.2

Because both K (in the definition of ρ_0) and the random Hamiltonian H_ω are bilinear in a^+, a (of the form $\int du_1 du_2 k(u_1,u_2)a^+_{u_1}a_{u_2}$), the same is true for

$$K(t) := e^{itH_\omega}Ke^{-itH_\omega}, \tag{5.1}$$

with probability 1. Therefore,

$$\rho_t(\cdot) = \frac{1}{Z_K}\mathrm{Tr}(e^{-K(t)}(\cdot)) \tag{5.2}$$

is quasifree with probability 1 (see, for instance, [3]). Thus, for $r,s\in\mathbb{N}$,

$$\rho_t(a^+(f_1)\cdots a^+(f_r)a(g_1)\cdots a(g_s)) = \delta_{r,s}\det\Big[\rho_t(a^+(f_j)a(g_\ell))\Big]_{j,\ell=1}^r, \tag{5.3}$$

where $f_j, g_\ell \in \mathcal{S}(\mathbb{T}^d)$ belong to the Schwartz class. In particular, we can set $r = s$.

We expand the determinant into

$$\det\Big[\rho_t(a^+(f_j)a(g_\ell))\Big]_{j,\ell=1}^r = \sum_{s\in S_r}(-1)^{\mathrm{sign}(s)}\prod_{j=1}^r \rho_t(a^+(f_j)a(g_{s(j)})), \tag{5.4}$$

where S_r is the symmetric group of degree r. We claim that for $T>0$ and $t=\frac{T}{\eta^2}$, and any choice of $f_j, g_\ell \in \mathcal{S}(\mathbb{T}^d)$,

$$\lim_{L\to\infty}\left|\mathbb{E}\left[\prod_{j=1}^r \rho_{T/\eta^2}(a^+(f_j)a(g_{s(j)}))\right] - \prod_{j=1}^r \mathbb{E}[\rho_{T/\eta^2}(a^+(f_j)a(g_{s(j)}))]\right| < \eta^\delta, \tag{5.5}$$

for a constant $\delta>0$ independent of r, $s\in S_r$, η, and T, and for $\eta>0$ sufficiently small. This immediately implies that, for every fixed $r<\infty$,

$$\lim_{L\to\infty}\left|\mathbb{E}\Big[\rho_{T/\eta^2}(a^+(f_1)\cdots a^+(f_r)a(g_1)\cdots a(g_r))\Big] - \det\Big[\mathbb{E}[\rho_{T/\eta^2}(a^+(f_j)a(g_\ell))]\Big]_{j,\ell=1}^r\right| < r!\eta^\delta \tag{5.6}$$

converges to zero as $\eta\to 0$.

This implies that

$$\Omega_T^{(2r)}(f_1,\dots,f_r;g_1,\dots,g_r) := \lim_{\eta\to 0}\lim_{L\to\infty}\mathbb{E}[\rho_{T/\eta^2}(a^+(f_1)\cdots a^+(f_r)a(g_1)\cdots a(g_r))] \tag{5.7}$$

is quasifree, i.e.,

$$\Omega_T^{(2r)}(f_1,\dots,f_r;g_1,\dots,g_r) = \det\big[\Omega_T^{(2)}(f_i;g_j)\big]_{1\le i,j\le r}, \tag{5.8}$$

where

$$\Omega_T^{(2)}(f;g) = \int dp\, F_T(p)\overline{f(p)}g(p). \tag{5.9}$$

The function $F_T(p)$ solves the linear Boltzmann equation with initial condition $F_0(p)$, as given in Theorem 3.1.

5.1 Proof of (5.5)

The inequality (5.5) follows from a straightforward application of the main results in [8] where we refer for details. In this section, we shall only outline the strategy. The expectation

$$\lim_{L\to\infty}\mathbb{E}\left[\prod_{j=1}^r \rho_t(a^+(f_j)a(g_{s(j)}))\right] \tag{5.10}$$

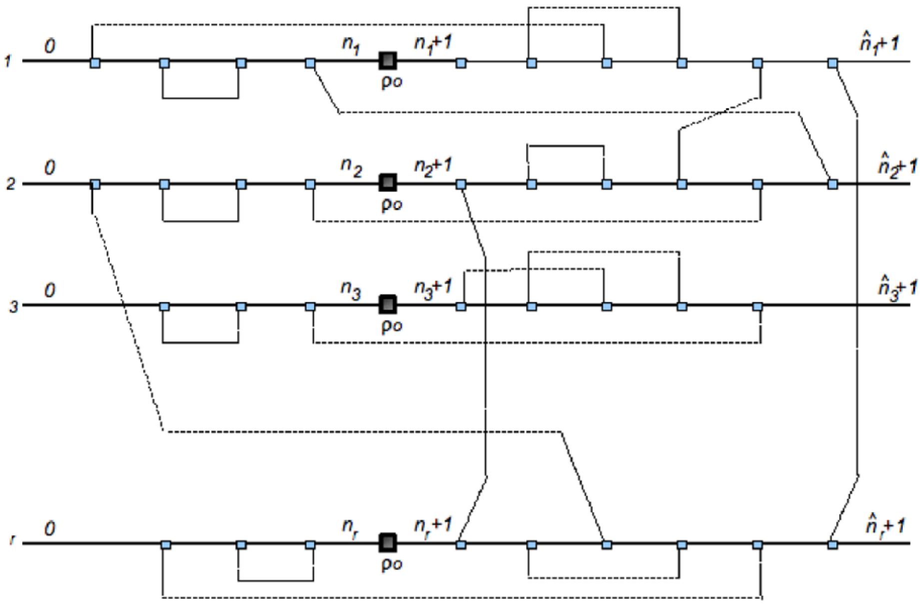

Fig. 2 Order r Feynman graph. The particle line indexed by $j = 3$ is disconnected

can be represented by a graph expansion as follows. We expand each of the factors

$$\rho_t(a^+(f_j)a(g_{s(j)})) = \sum_{n,\widetilde{n}=1}^{N+1} \int dp J(p) \overline{f_{j,t}^{(n)}(p)} g_{s(j),t}^{(\widetilde{n})}(p) \tag{5.11}$$

separately into a truncated Duhamel series of level N, using the same definitions as in (4.16). For the remainder term (where at least one of the indices $n, \widetilde{n}$ equals $N+1$), we subdivide the time integration interval $[0,t]$ into κ pieces of length $\frac{t}{\kappa}$.

For the expectation (5.10), we introduce the following extension of the classes of Feynman graphs discussed for the proof of Theorem 3.1, see also Fig. 2. For $r > 1$, we consider r particle lines parallel to one another, each containing a distinguished ρ_0-vertex separating it into a left and a right part. Enumerating them from 1 to r, the j-th particle line contains n_j interaction vertices on the left of the ρ_0-vertex, and $\widetilde{n}_j$ interaction vertices on its right. We note that for $r > 1$, only $\sum_{j=1}^r (n_j + \widetilde{n}_j)$ has to be an even number, but not each individual

$$\widehat{n}_j := n_j + \widetilde{n}_j. \tag{5.12}$$

On the j-th interaction line, we label the propagator lines by momentum variables $u_0^{(j)}, \dots, u_{\widehat{n}_j+1}^{(j)}$, with indices increasing from the left.

A *contraction graph* of degree $\{(n_j, \widetilde{n}_j)\}_{j=1}^r$ is obtained by connecting pairs of interaction vertices by contraction lines. We denote the set of contraction graphs of degree $\{(n_j, \widetilde{n}_j)\}_{j=1}^r$ by $\Gamma_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}$. If the ℓ-th vertex on the j-th particle line is connected by a contraction line to the ℓ'-th vertex on the j'-th particle line, we write

$$(j;\ell) \sim_\pi (j';\ell'). \tag{5.13}$$

To a graph $\pi \in \Gamma_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}$, we associate the *Feynman amplitude*

$$\begin{aligned}
&\mathrm{Amp}_\pi(\{f_j, g_{s(j)}\}; \eta; T) \\
&:= \eta^{2\sum_{1\le j\le r}(n_j+\widetilde{n}_j)} e^{2r\epsilon t} \prod_{j=1}^r \int d\alpha_j d\widetilde{\alpha}_j e^{it(\alpha_j-\widetilde{\alpha}_j)} \\
&\quad \times \int du_0^{(j)} \cdots du_{\widetilde{n}_j+1}^{(j)} \overline{f_j(u_0^{(j)})} g_{s(j)}(u_{\widetilde{n}_j+1}^{(j)}) J(u_{n_j}^{(j)}) \delta(u_{n_j}^{(j)} - u_{n_j+1}^{(j)}) \\
&\quad \times \delta_\pi(\{u_i^{(j)}\}_{i=0}^{\widehat{n}_j+1}) \prod_{\ell=0}^{n_j} \frac{1}{E(u_\ell^{(j)}) - \alpha_j - i\epsilon} \prod_{\ell'=n_j+2}^{\widehat{n}_j} \frac{1}{E(u_{\ell'}^{(j)}) - \widetilde{\alpha}_j + i\epsilon},
\end{aligned} \tag{5.14}$$

where

$$\epsilon = \frac{1}{t} = \frac{\eta^2}{T} \tag{5.15}$$

for $T > 0$. The delta distribution

$$\delta_\pi(\{u_j^{(j)}\}_{j=0}^{\widehat{n}_j+1}) = \prod_{(j;\ell)\sim_\pi(j';\ell')} \delta(u_\ell^{(j)} - u_{\ell-1}^{(j)} - (u_{\ell'}^{(j')} - u_{\ell'-1}^{(j')})) \tag{5.16}$$

is the product of delta distributions associated to all contraction lines in π.

5.1.1 Completely Disconnected Graphs

The subclass

$$\Gamma^{disc}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r} \subset \Gamma_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r} \tag{5.17}$$

of *completely disconnected* graphs of degree $\{(n_j, \widetilde{n}_j)\}_{j=1}^r$ consists of those graphs in which contraction lines only connect interaction vertices on the same particle line.

It is clear that

$$\lim_{L\to\infty} \sum_{\substack{0\le n_j,\widetilde{n}_j\le N\\ j=1,\dots,r}} \sum_{\pi\in\Gamma^{disc}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}} \mathrm{Amp}_\pi(\{f_j, g_{s(j)}\}; \eta; T) \tag{5.18}$$

$$= \lim_{L\to\infty} \prod_{j=1}^r \sum_{n_j,\widetilde{n}_j=1}^N \mathbb{E}\Big[\int dp J(p) \overline{f_{j,T/\eta^2}(p)} g_{s(j),T/\eta^2}(p)\Big]$$

$$= \lim_{L\to\infty} \prod_{j=1}^r \big(\mathbb{E}[\rho_{T/\eta^2}(a^+(f_j)a(g_{s(j)}))] + O(\eta^\delta)\big), \tag{5.19}$$

according to our proof of Theorem 3.1. The term of order $O(\eta^\delta)$ accounts for the remainder term associated to the j-th particle line (i.e., the terms involving $\mathbb{E}[\rho_{T/\eta^2}^{(n_j,\widetilde{n}_j)}(p,q)]$ where at least one of the indices $n_j, \widetilde{n}_j$ equals N). Thus, for any fixed $r \in \mathbb{N}$, we obtain

$$\lim_{\eta\to 0}\lim_{L\to\infty} \sum_{\substack{0\le n_j,\widetilde{n}_j\le N\\ j=1,\dots,r}} \sum_{\pi\in\Gamma^{disc}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}} \mathrm{Amp}_\pi(\{f_j, g_{s(j)}\}; \eta; T)$$

$$= \prod_{j=1}^{r} \Omega_T^{(2)}(f_j; g_{s(j)}). \tag{5.20}$$

That is, the sum over completely disconnected graphs yields the corresponding product of averaged 2-point functions in the kinetic scaling limit.

5.1.2 Non-Disconnected Graphs

We refer to the complement of the set of completely disconnected graphs in $\Gamma_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}$,

$$\Gamma^{n-d}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r} := \Gamma_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r} \setminus \Gamma^{disc}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}, \tag{5.21}$$

as the set of *non-disconnected graphs*. It remains to prove that the sum over non-disconnected graphs, combined with the remainder terms, can be bounded by $O(\eta^\delta)$, for L sufficiently large.

The condition required in [8] for the estimate analogous to (5.5) to hold is that for the initial condition ϕ_0 (corresponding to the test functions f_j, g_ℓ in our case) of the random Schrödinger evolution studied in [8], a "concentration of singularity condition" is satisfied (that is, singularities in momentum space are not too much "spread out" in the limit $\eta \to 0$). It states that in frequency space $\mathbb{T}^d$,

$$\widehat{\phi}_0 = \widehat{\phi}_0^{(reg)} + \widehat{\phi}_0^{(sing)}, \tag{5.22}$$

where

$$\|\widehat{\phi}_0^{(reg)}\|_\infty < c \tag{5.23}$$

and

$$\| |\widehat{\phi}_0^{(sing)}| * |\widehat{\phi}_0^{(sing)}| \|_2 < c' \eta^{3/2} \tag{5.24}$$

are satisfied uniformly in L, as $L \to \infty$.

In the present case, we have to require that f_j, g_ℓ satisfy the concentration of singularity condition. This is, however, evidently fulfilled since f_j, g_ℓ are η-independent Schwartz class functions (in contrast, the initial states considered in [8] are of WKB type, and scale nontrivially with η).

It is proven in [8] that the amplitude of every non-disconnected graph with $n_j, \widetilde{n}_j \le N$ for $j = 1, \dots, r$, is bounded by

$$\sup_{\pi \in \Gamma^{n-d}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}} \left|\mathrm{Amp}_\pi(\{f_j, g_{s(j)}\}; \eta; T)\right| \tag{5.25}$$

$$< \epsilon^{1/5} \left(c\eta^2 \epsilon^{-1} \log \frac{1}{\epsilon} \right)^{\frac{r}{2} \sum_{j=1}^r \widehat{n}_j} \left(\log \frac{1}{\epsilon} \right)^{4r}, \tag{5.26}$$

where we recall that $\epsilon = \frac{1}{t} = \frac{\eta^2}{T}$ for $T > 0$. This key estimate is a factor $\epsilon^{1/5}$ smaller than the bound on the sum of disconnected graphs; this improvement is obtained from exploiting the existence of at least one contraction line that connects two different particle lines; see [8].

The number of non-disconnected graphs is bounded by

$$\left|\Gamma^{n-d}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}\right| \leq \left(\sum_{j=1}^{r}\widehat{n}_j\right)! \leq (2rN)!, \tag{5.27}$$

where $\widehat{n}_j = n_j + \widetilde{n}_j$. Therefore, the sum of amplitudes of all non-disconnected graphs with $0 \leq n_j, \widetilde{n}_j \leq N$ is bounded by

$$\sum_{1\leq j\leq r}\ \sum_{0\leq n_j,\widetilde{n}_j\leq N}\ \sum_{\pi\in\Gamma^{n-d}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}} \left|\mathrm{Amp}_\pi(\{f_j, g_{s(j)}\};\eta;T)\right| \tag{5.28}$$

$$\leq ((2rN)!)^2\epsilon^{1/5}\left(c\eta^2\epsilon^{-1}\log\frac{1}{\epsilon}\right)^{rN}\left(\log\frac{1}{\epsilon}\right)^{4r}. \tag{5.29}$$

Here we have estimated the sum over pairs $0 \leq n_j, \widetilde{n}_j \leq N$, $1 \leq j \leq r$, by another factor $(2rN)!$, since $\#\{(n_j,\widetilde{n}_j)\}_{j=1}^r | \sum_j \widehat{n}_j = m\} \leq m!$.

5.1.3 Duhamel Remainder Term

In case at least one of the indices n_j or $\widetilde{n}_j$ equals $N+1$, the following argument can be applied. Clearly, from a Hölder estimate of the form $\|h_1\cdots h_s\|_1 \leq \|h_1\|_s\cdots\|h_s\|_s$ with respect to $\mathbb{E}$, we have

$$\left|\mathbb{E}\left[\prod_{j=1}^{r}\rho_t^{(n_j,\widetilde{n}_j)}(f;g)\right]\right| \leq \prod_{j=1}^{r}\mathbb{E}[|\rho_t^{(n_j,\widetilde{n}_j)}(f;g)|^{2r}]^{\frac{1}{2r}}. \tag{5.30}$$

Here, we have used an exponent $2r$ instead of r because then, even for r odd, an absolute value of the form $|z|^{2r}$ can be replaced by a product of the form $\overline{z}^r z^r$, where $z \in \mathbb{C}$.

We make a choice of constants

$$t = \frac{1}{\epsilon} = \frac{T}{\eta^2},$$
$$N = \frac{\log\frac{1}{\epsilon}}{10r\log\log\frac{1}{\epsilon}}, \tag{5.31}$$
$$\kappa = \left(\log\frac{1}{\epsilon}\right)^{15r},$$

similarly as in Sect. 4.7 of the proof of Theorem 3.1.

If n_j or $\widetilde{n}_j$ equals $N+1$, we can use the bounds (4.52) and (4.53).

If both $n, \widetilde{n} \leq N$, we use the a priori bound

$$\sum_{n+\widetilde{n}=2\bar{n}}\ \sum_{\pi\in\Gamma_{n,\widetilde{n}}}\lim_{L\to\infty}\mathbb{E}[|\rho_t^{(n,\widetilde{n})}(f;g)|^{2r}]^{\frac{1}{2r}} \tag{5.32}$$

$$< \left[\sum_{\ell=0}^{2r}\binom{2r}{\ell}\left(\frac{(c\eta^2\epsilon^{-1})^{\bar{n}}}{(\bar{n}!)^{1/2}}\right)^{\ell}\right.$$

$$\times \epsilon^{1/5}((2r-\ell)\bar{n})!\left(\left(\log\frac{1}{\epsilon}\right)^4\left(c\eta^2\epsilon^{-1}\log\frac{1}{\epsilon}\right)^{\bar{n}}\right)^{2r-\ell}\Bigg]^{\frac{1}{2r}}$$

$$< \frac{(c\eta^2\epsilon^{-1})^{\bar{n}}}{(\bar{n}!)^{1/2}} + \eta^{\frac{1}{10}}. \tag{5.33}$$

The factor $\frac{(c\eta^2\epsilon^{-1})^{\ell\bar{n}}}{(\bar{n}!)^{\ell}}$ in $[\cdots]$ accounts for ℓ basic ladders on ℓ copies of $\Gamma^{disc}_{n,\widetilde{n}}$, while the remaining factor accounts for all other (not necessarily non-disconnected) contractions on the remaining $2r-\ell$ particle lines; for details, see [7, 8, 11].

Let us without any loss of generality assume that $n_1 = N+1$. Then, keeping n_1 fixed and summing over the remaining indices $\widetilde{n}_1$ and $n_j, \widetilde{n}_j$, with $j = 2, \dots, r$, we find

$$\sum_{\substack{0\le n_2, n_j, \widetilde{n}_j \le N+1 \\ j=2,\dots,r}} \prod_{j=1}^{r} \mathbb{E}[|\rho_t^{(n_j,\widetilde{n}_j)}(f;g)|^{2r}]^{\frac{1}{2r}}$$

$$< \eta^{\frac{1}{15}}\left[\sum_{\bar{n}=0}^{N}\frac{(c\eta^2\epsilon^{-1})^{\bar{n}}}{(\bar{n}!)^{1/2}} + \eta^{\frac{1}{10}}\right]^{2r-1}, \tag{5.34}$$

where the factor $\eta^{\frac{1}{15}}$ accounts for the remainder term indexed by $n_1 = N+1$. We conclude that the sum over all terms (5.30) which contain at least one n_j or $\widetilde{n}_j$ equalling $N+1$ (i.e., which contain at least one Duhamel remainder term) can be bounded by

$$C^r \eta^{\frac{1}{15}} \tag{5.35}$$

for a constant C independent of η and r.

Combined with

$$(5.28) < \eta^{\frac{1}{20}}, \tag{5.36}$$

which one easily verifies, this completes the proof of Theorem 3.2. For more details addressing the arguments outlined here, we refer to [8].

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